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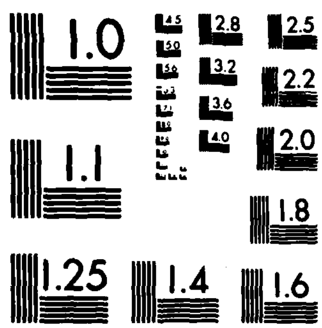

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A Free Electron Laser with a Rotating Quadrupole Wiggler

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A new rotating quadrupole wiggler configuration for free-electron lasers is proposed. The stability analyses of the particles trajectories in a continuously rotating quadrupole magnetic field and self-fields of a solid beam is performed. The resulting orbit equations are solved exactly and it is shown that high-current beams can be confined. To analyze the potential of this wiggler for free electron lasers, a set of nonlinear orbit equations is derived which average over the fast time variation in both the wiggler and radiation fields. By integrating these equations, the linearized single particle gain of the free electron laser is calculated. It is shown that at comparable wiggler strengths, the rotating quadrupole wiggler and conventional wiggler give similar gains.

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CONTENTS

A FREE ELECTRON LASER WITH A ROTATING QUADRUPOLE WIGGLER

I. Introduction

Free electron lasers[1,2] are devices which are designed to produce coherent radiation by passing a relativistic electron beam through some sort of spatially periodic perturbation.[3] Commonly, a periodic magnetic field, called a wiggler, is used.

In general, the wiggler field varies transversely as well as axially. However, in most experiments the initial beam radius, R_b is a small fraction of the wiggler period, λ_w and the transverse variations of the wiggler are neglected. For example, the field of a periodic helical winding of two wires around a drift tube to lowest order in $\epsilon = k_w R_b$ $(k_w = 2\pi/\lambda_w)$ is commonly represented by a simple periodic function of axial distance alone

$$\underline{B}_d = B_{od}[\cos (k_d z)\underline{i}_x + \sin (k_d z,)\underline{i}_y]. \qquad (1)$$

We provided the notation of this field with subscript "d" in order to indicate that such a field has a dipole character.

Generally, a second field in the axial direction is added to confine the beam of particles against their natural repulsion.

In principle, one can generate a periodic magnetic field by a helical winding of 2ℓ wires $(\ell = 1,2...)$. A magnetic field of this kind exhibits helical symmetry and it is convenient to describe it by means of a scalar potential that satisfied Laplace's equation.[4] This potential depends only on dimensionless coordinates $\rho = \alpha_w r$ and $u_w = \theta_w - \alpha_w z$ and has a form

$$\Phi_\ell = \frac{1}{\alpha_w} I_\ell(\ell\rho) \sin \ell u_w. \qquad (2)$$

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Where I_ℓ is a modified Bessel function of order ℓ, the pitch of the helix α_w is defined by,

$$\alpha_w = k_w/\ell \tag{3}$$

and θ is the polar angle. By definition the components of the corresponding magnetic field can be deduced from

$$\underline{B} = \text{grad } \Phi_\ell . \tag{4}$$

Consider now that the periodic magnetic field is generated by a helical winding of four wires as shown in Fig. 1 (the current flow is the same as in a stellerator configuration).

We expand I_2 to second order in ε and from Eq. (4) we obtain

$$B_{r,q} = B_{oq}\, k_q r \sin\,(2\theta - k_q z)$$

$$B_{\theta,q} = B_{oq}\, k_q r \cos\,(2\theta - k_q z)$$

$$B_{zq} = -B_{oq}\left(\frac{k_q r}{2}\right)^2 \cos\,(2\theta - k_q z) \tag{5}$$

where B_{oq} is the value of the magnetic field at $r = \lambda_q/2\pi$, and the subscript q indicates that the field is quadrupole in nature. To lowest order in ε from Eqs. (5) we obtain for the magnetic field components in cartesian coordinates

$$B_{xq} = B_{oq} k_q \, (y \cos k_q z - x \sin k_q z)$$

$$B_{yq} = B_{oq} \, k_q \, (x \cos k_q z + y \sin k_q z)$$

$$B_{zq} = 0. \qquad (6)$$

One can show that the field lines of the above magnetic field, Eqs. (6), are similar to the line forces of a quadrupole magnet continuously rotating along the z axis. Quadrupole fields are known in accelerator physics for their focusing ability of the beam current.[5]

In this paper, we investigate the feasibility of using a quadrupole magnetic field as a wiggler in free elelctron laser devices as well as a focusing field to confine the beam. We assume, that if $R_b/\lambda_q \ll 1$, the actual field can be good represented by Eqs. (6). Because of the strong focusing property of the field, one might expect that large beam currents could stably propagate. However, because the field is zero on the axis, the free electron laser mechanism might be weaker. Here we investigate these issues.

II. Particle Orbits and Stability Analyses

The physical model we develop will consider a relativistic non neutral electron beam with radius R_b propagating in the quadrupole magnetic field given by Eqs. (6). We assume that the beam is solid and has a uniform density $n_b(r) = n_o$. The expressions for the equilibrium self-electric and self-magnetic fields associated with the beam space charge and axial current $j_b = -en_o v_b$ (v_b-beam axial velocity) are given by,

$$E_{xs} = - 2\pi|e|n_o x$$

$$E_{ys} = - 2\pi|e|n_o y$$

$$B_{xs} = 2\pi|e|n_o\beta_b y,$$

and

$$B_{ys} = - 2\pi|e|n_o\beta_b x, \tag{7}$$

for $0 < r = (x^2 + y^2)^{1/2} < R_b$

where $|e|$ is the electron charge and $\beta_b = v_b/c$. The electron orbits within the beam are determined from the equations of motion

$$\frac{d}{dt}(\gamma v_x) = \frac{1}{2}\,\omega_b^2(1 - \beta_z\beta_b)x + v_z\Omega_q k_q(x \cos k_q z + y \sin k_q z)$$

$$\frac{d}{dt}(\gamma v_y) = \frac{1}{2}\,\omega_b^2(1 - \beta_z\beta_b)y + v_z\Omega_q k_q(x \sin k_q z - y \cos k_q z)$$

$$\frac{d}{dt}(\gamma v_z) = \frac{1}{2}\,\omega_b^2\beta_b(x\beta_x + y\beta_y) + \Omega_q k_q\left[v_y(y \cos k_y z - x \sin k_q z)\right.$$

$$\left. - v_x(x \cos k_q z + y \sin k_q z)\right] \tag{8}$$

where $\omega_b^2 = 4\pi n_o e^2/m$ is the beam plasma frequency and $\Omega_q = |e|B_{oq}/mc$ is the cyclotron frequency. From the above equations and the assumption of this analysis $(k_q x\ (k_q y) = \varepsilon \ll 1)$ is evident that $v_x/v_z \sim \varepsilon$, and $v_y/v_z \sim \varepsilon$,

therefore to lowest order in ε the system (8) reduces to a simple system of equations describing the transverse motion of an electron.

$$v_z \frac{dv_x}{dz} = \frac{1}{2}\frac{\omega_b^2}{\gamma_o^3} x + v_z \frac{\Omega_q}{\gamma_o} k_q (x \cos k_q z + y \sin k_q z)$$

$$v_z \frac{dv_y}{dz} = \frac{1}{2}\frac{\omega_b^2}{\gamma_o^3} y + v_z \frac{\Omega_q}{\gamma_o} k_q (x \sin k_q z - y \cos k_q z) \qquad (9)$$

where $\gamma = \gamma_o \approx$ const. and $\beta_b = \beta_z \approx$ const., hence, $\frac{d}{dt} = v_z \frac{d}{dz}$.

We introduce dimensionless variables; the distance is multiplied by k_q and frequency is divided by $(k_q c)$. Then the normalized Eqs. (9) have a form

$$\frac{d^2x}{dz^2} = \alpha\,(x \cos z + y \sin z) + \delta x$$

$$\frac{d^2y}{dz^2} = \alpha\,(x \sin z - y \cos z) + \delta y \qquad (10)$$

where

$$\alpha \equiv \Omega_q/(\gamma_o \beta_z k_q c),$$

$$\delta \equiv \omega_b^2/(2\gamma_o^3 k_q^2 c^2) \qquad (11)$$

α measures the strength of the quadrupole fields and δ measures the beam density. Although we did not change the notation for x, y and z, one should regard these variables as dimensionless.

In order to solve the system of Eqs. (10) it is convenient to introduce new variables u and v related to x and y by the following transformation

$$x = u \cos(z/2) + v \sin(z/2)$$

$$y = u \sin(z/2) - v \cos(z/2). \tag{12}$$

In the new variables, Eqs. (10) becomes a system of homogeneous equations with constant coefficients, namely

$$\frac{d^2u}{dz^2} + \frac{dv}{dz} - (\alpha + \frac{1}{4} + \delta)\, u = 0$$

$$\frac{d^2v}{dz^2} - \frac{du}{dz} + (\alpha - \frac{1}{4} + \delta)\, v = 0. \tag{13}$$

The solution of this system of equations has, in general, the following form

$$u = A \exp ikz$$

and

$$v = B \exp ikz. \tag{14}$$

Inserting this solution into Eqs. (13), we obtain a system of two algebraic equations to determine the coefficients A and B. Non-triviality of the solution of that algebraic system provides the equation for k

$$(k^2 + \frac{1}{4} + \delta)^2 - \alpha^2 - k^2 = 0. \tag{15}$$

The four independent solutions of the bi-quadratic equation, (15), are given by

$$k_s = \pm \left[\left(\frac{1}{4} - \delta\right) \pm [\alpha^2 - \delta]^{1/2} \right]^{1/2}. \qquad (16)$$

If one of the k_s has a non-zero imaginary part the solutions (14) become unstable. Therefore, for Im $k_s = 0$, the following conditions are required

$$\sqrt{\delta} < \alpha < (\delta + 1/4) \qquad (17a)$$

or equivalently,

$$\omega_b^2 < \frac{2\gamma_o \Omega_q^2}{\beta_o^2} < \frac{(2\omega_b^2 + k_q^2 c^2 \gamma_o^3)^2}{8\gamma_o^3 k_q^2 c^2}. \qquad (17b)$$

In the limit of negligibly small self fields ($\delta \to 0$) the stability conditions (17) become

$$\alpha < \frac{1}{4}.$$

Equation (17) indicates that a quadrupole magnetic field should be strong enough to overcome the repulsive forces due to self fields of the beam, but not too strong.

The stable region in the parameter space α^2 and δ is shown in Fig. 2 between the two solid lines, $\alpha^2 = \delta$ and $\alpha^2 = (\delta + 1/4)^2$. It is convenient to relate the parameter δ to the beam current. The relation is

$$I_b/I_a = \delta/2 \; (\beta_z\gamma_o)^2 \; (k_q R_b)^2$$

where $I_a = mc^3/e\beta_z\gamma_o$ is the Alfvén-Lawson current.

From Eq. (17) it follows that the theoretical maximum for δ in the stable region is determined by $\delta = (\delta + 1/4)^2$ and is equal to 1/4. Therefore, in principle, the maximum electron beam current which is able to propagate in a quadrupole magnetic field is

$$I_{b,max} = \frac{1}{8} \gamma_o^2 (k_q R_b)^2 I_a. \tag{18}$$

For example, if $\gamma_o = 4$, $k_q R_b \approx \frac{1}{4}$, $\lambda_q \approx 3$ cm, $I_{b,max} \approx 8.5$ kA, which require $B_{oq} \approx 3.5$ kG.

It is also interesting to compare the maximum beam density which can be confined by a rotating quadrupole field with that which can be confined by a uniform axial magnetic field. It is not difficult to show in the latter case that the condition for stable orbits is

$$\omega_b^2 < \frac{\gamma_o \Omega_c^2}{2} \tag{19}$$

where Ω_c is the cyclotron frequency in the axial field. Comparing Eq. (19) with Eq. (17b), it is apparent that the quadrupole field with $\Omega_q = \Omega_c$ can confine four times the beam current.

III. <u>Single Particle Analyses of a Free Electron Laser with Quadrupole Wiggler</u>

In the previous section we investigated the stability of the electron orbits in a relativistic electron beam propagating in a quadrupole magnetic field, Eqs. (6). Here, we will show that such a field can be used as a wiggler in a free electron laser device to produce coherent radiation with wavelength $\lambda_r \approx \lambda_q / 2\gamma_o^2$. Henceforth, a subscript r denotes the radiation field.

In our analyses we will use a single particle approach to describe the interaction of a relativistic electron with the periodic quadrupole magnetic field and a plane circulary polarized electromagnetic wave. This is analogous to the calculation of Colson[6] of free electron laser gain in a conventional wiggler (Eq. (1)), oscillator configuration with a specified radiation field. The radiation is represented by a circularly polarized electromagnetic wave

$$\underline{E}_r = E_o\left[\cos(k_r z - \omega_r t + \phi)i_x - \sin(k_r z - \omega_r t + \phi)\ i_y\right]$$

$$\underline{B}_r = i_z x\ \underline{E}_r \tag{20}$$

with i_x, i_y and i_z being unit vectors in x, y and z direction, respectively, and $\omega_r = k_r c$.

The equations of motion for the relativistic electron in the combined electromagnetic wave (20) and pump wave (wiggler) (6) are given by

$$\frac{d}{d\tau}\gamma\frac{dx}{d\tau} = -e_{rx}(1-\beta_z) + \beta_z b_{qy} \tag{21a}$$

$$\frac{d}{d\tau}\gamma\frac{dy}{d\tau} = -e_{ry}(1-\beta_z) - \beta_z b_{qx} \tag{21b}$$

$$\frac{d}{d\tau}\gamma\frac{dz}{d\tau} = -(\beta_y(b_{qx} - \rho_{ry}) - \beta_x(b_{qy} + \rho_{rx}). \tag{21c}$$

As before, we kept only terms of lowest order in ε and we introduced normalized variables. We also denoted

$$\underline{e}_r = e_{ra}\left[\cos\left[\kappa(z-\tau)+\phi\right]i_x - \sin\left[\kappa(z-\tau)+\phi\right]i_y\right. \tag{22}$$

$$\underline{b}_q = b_{qa}[(y \cos z - x \sin z)i_x + (x \cos z + y \sin z)i_y] \qquad (23)$$

where

$$e_{ra} = |e|E_o/(mck_q c)$$

$$b_{qa} = \Omega_q/(k_q c) \qquad (24)$$

and $\kappa = k_r/k_q$, $\tau = (k_q c)t$ x, y and z are also dimensionless (note that $\kappa \approx 2\gamma_o^2 \gg 1$).

The orbit equations (21) describing the motion transverse and parallel to the electron stream are coupled, due to the x and y dependence of the pump field. This fact makes direct integration difficult, even making a linear approximation. The principal complication is that the orbit has two very different frequencies, a forced oscillation at the doppler shifted pump or radiation frequency, and a slower evolution at the beat frequency. To proceed we will average over the fast oscillation and calculate the motion only on the slow time scale.

IV. Multiple-Time Scale Analyses

In order to find the solution of the set of the nonlinear coupled equation (21), we will utilize multiple-time scale analyses.[7] This gives the same result as a similar derivation making use of the covariant structure of the equations of motion.[8] We introduce two time scales, one fast - τ_o, another slow - τ_1. The fast time scale τ_o is associated with the pump field "frequency" - $(k_q c)^{-1}$, and the slow time scale τ_1 is proportional to $k_q v_\perp$, which in turn is proportional to Ω_q^{-1} and $(|e|E_o/mc)^{-1}$. Also, in the slow

time equations we retain as a low frequency term the beat frequency $\omega_r - (k_r + k_q)v_z$.

We assume that variations on fast time scale are periodic, and both variables τ_o and τ_1 are treated as independent variables. To introduce the difference between fast and slow time scales we consider a formal procedure consisting of assuming a perturbation expansion of the form

$$x(\tau_o,\tau_1) = x_o(\tau_o,\tau_1) + x_1(\tau_o,\tau_1) + x_2(\tau_o,\tau_1) + \ldots$$

and

$$\frac{d}{dt} = \frac{\partial}{\partial\tau_o} + \frac{\partial}{\partial\tau_1} \tag{25}$$

from which follows

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial\tau_o^{\,2}} + 2\,\frac{\partial^2}{\partial\tau_o\partial\tau_1} + \frac{\partial^2}{\partial\tau_1^2}$$

where we denoted the order of the quantities with a subscript o,1,..., hence x_o and $\frac{\partial}{\partial\tau_o}$, x_1 and $\frac{\partial}{\partial\tau_1}$ and x_2 and $\frac{\partial^2}{\partial\tau_1^{\,2}}$ are zero, first, and second order quantities, respectively.

Substituting Eqs. (25) into Eqs. (21) for zero order equations we obtain

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial x_o}{\partial\tau_o} = 0$$

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial y_o}{\partial\tau_o} = 0$$

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial z_o}{\partial\tau_o} = 0. \tag{26}$$

The solutions to Eqs. (26) are

$$x_o(\tau_1,\tau_o) = x_o(\tau_1)$$

$$y_o(\tau_1,\tau_o) = y_o(\tau_1)$$

$$z_o(\tau_1,\tau_o) = z_o(\tau_1) + \beta_o(\tau_1)\tau_o \tag{27}$$

where $\partial z_o/\partial\tau_o = \beta_o(\tau_1)$ and $\partial x_o/\partial\tau_o = \partial y_o/\partial\tau_o = 0$, since we assumed that zero order transverse motion doesn't exist. The quantity $\beta_o(\tau_1)$ as expressed in Eq. (27) is undefined because any velocity which is nearly equal to the particle velocity will render $z_o(\tau_1)$ and $\beta_o(\tau_1)$ slowly varying functions of time. To be more specific, we define $\beta_o(\tau_1) = v_o/c$ in terms of the resonant velocity, so that

$$v_o = \frac{\omega_r}{k_r + k_q}. \tag{28}$$

Since no quantity on the right-hand side of Eq. (28) depends on time, $\beta_o(\tau_1)$ is independent of τ_1. We will now show that this is consistent.

The first order part for Eq. (21c) is given by

$$\frac{\partial}{\partial\tau_o}\left(\gamma_o \frac{\partial z_1}{\partial\tau_o}\right) + \frac{\partial}{\partial\tau_1}(\gamma_o\beta_o) + \frac{\partial}{\partial\tau_o}(\gamma_1\beta_o) + \frac{\partial}{\partial\tau_o}\left(\gamma_o \frac{\partial z_o}{\partial\tau_1}\right) = 0. \tag{29}$$

Upon averaging over the fast time scale we have,

$$\frac{\partial}{\partial\tau_1}(\gamma_o\beta_o) = 0. \tag{30}$$

Note that γ_o depends only on β_o, therefore Eq. (30) requires

$$\beta_o\,(\tau_1) = \text{const.}$$

Thus our assumption that β_o is independent of τ_1 is consistent. Making use of this fact we return to Eq. (29) and integrate it once in τ_o. The resulting equation is given by

$$\gamma_o\,\left(\frac{\partial z_1}{\partial \tau_o} + \frac{\partial z_o}{\partial \tau_1}\right) + \gamma_1 \beta_o = C(\tau_1) \tag{31}$$

where $C(\tau_1)$ is a constant of the integration.

Expanding γ to the first order and utilizing Eq. (27), we find that

$$\gamma_1 = \gamma_o^3 \beta_o \left(\frac{\partial z_o}{\partial \tau_1} + \frac{\partial z_1}{\partial \tau_o}\right). \tag{32}$$

Substituting it into Eq. (31) we obtain

$$\frac{\partial z_1}{\partial \tau_o} + \frac{\partial z_o}{\partial \tau_1} - \frac{C(\tau_1)}{\gamma_o^3} = 0. \tag{33}$$

The last two terms do not depend on τ_o, therefore by averaging Eq. (33) over τ_o we obtain

$$\frac{\partial z_o}{\partial \tau_1} = \frac{C(\tau_1)}{\gamma_o^3}.$$

Substituting this result back into Eq. (33) we find that

$$\frac{\partial z_1}{\partial \tau_o} = 0,$$

follows, that

$$\gamma_1(\tau_1) = \gamma_o^3 \beta_o \beta_{z,1}(\tau_1) \tag{34}$$

where $\partial z_o/\partial\tau_1 = \beta_{z,1}(\tau_1)$.

Consider now the first order transverse motion. The left-hand side of the Eq. (21a) becomes

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial x_o}{\partial\tau_o} + \frac{\partial}{\partial\tau_o}\{\gamma_o\frac{\partial x_o}{\partial\tau_1} + \gamma_1\frac{\partial x_o}{\partial\tau_o}\} + \frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial x_1}{\partial\tau_o}.$$

Since x_o does not depend on τ_o, the above expression reduces to

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial x_1}{\partial\tau_o}, \tag{35a}$$

and correspondingly, for left-hand side of Eq. (21b)

$$\frac{\partial}{\partial\tau_o}\gamma_o\frac{\partial y_1}{\partial\tau_o}. \tag{35b}$$

The radiation and wiggler field expressions have the following form

$$\underset{\sim}{e}_{r,1} = e_{ra}[\cos\{\kappa\{(\beta_o - 1)\tau_o + z_o\} + \phi\}i_x$$

$$- \sin\{\kappa\{(\beta_o - 1)\tau_o + z_o\} + \phi\}i_y,] \tag{36}$$

$$\underline{b}_{q,1} = b_{qa} \{[y_o \cos (z_o + \beta_o\tau_o) - x_o \sin (z_o + \beta_o\tau_o)]i_x$$

$$+ [x_o \cos (z_o + \beta_o\tau_o) + y_o \sin (z_o + \beta_o\tau_o)]i_y\}. \quad (37)$$

Combining Eqs. (21), (35)-(37) we obtain first order equations for the transverse motion

$$\frac{\partial^2 x_1}{\partial\tau_o^2} = -\frac{e_{ra}}{\gamma_o}(1 - \beta_o) \cos [\kappa\{(\beta_o - 1)\tau_o + z_o\} + \phi]$$

$$+ \frac{\beta_o b_{qa}}{\gamma_o}[x_o \cos (z_o + \beta_o\tau_o) + y_o \sin (z_o + \beta_o\tau_o)] \quad (38)$$

and

$$\frac{\partial^2 y_1}{\partial\tau_o^2} = \frac{e_{ra}}{\gamma_o}(1 - \beta_o) \sin [\kappa(\beta_o - 1)\tau_o + \kappa z_o + \phi]$$

$$- \frac{\beta_o b_{qa}}{\gamma_o}[y_o \cos (z_o + \beta_o\tau_o) - x_o \sin (z_o + \beta_o\tau_o)]. \quad (39)$$

Integrating Eqs. (38) - (39) one finds

$$x_1(\tau_o,\tau_1) = \frac{e_{ra}}{\gamma_o(1 - \beta_o)\kappa^2} \cos [\kappa\{(\beta_o - 1)\tau_o + z_o\} + \phi]$$

$$- \frac{b_{qa}}{\gamma_o\beta_o}[x_o \cos (z_o + \beta_o\tau_o) + y_o \sin (z_o + \beta_o\tau_o)] \quad (40)$$

$$y_1(\tau_o,\tau_1) = -\frac{e_{ra}}{\gamma_o(1 - \beta_o)\kappa^2} \sin [\kappa\{(\beta_o - 1)\tau_o + z_o\} + \phi]$$

$$+ \frac{b_{qa}}{\gamma_o\beta_o}[y_o \cos (z_o + \beta_o\tau_o) - x_o \sin (z_o + \beta_o\tau_o)]. \quad (41)$$

The dependence of the right-hand side on τ_1 is through z_o which is a function of τ_1. We proceed with the multi-time expansion procedure. To do so, we write the second order equations for Eqs. (21). The expressions for the wiggler field, evaluated along the first order orbit, and therefore correct to second order are given by

$$b_{qx} = b_{qa}\lfloor y_o \cos(z_o + \beta_o\tau_o) + y_1 \cos(z_o + \beta_o\tau_o) - z_1 y_o \sin(z_o + \beta_o\tau_o)$$

$$- x_o \sin(z_o + \beta_o\tau_o) - x_1 \sin(z_o + \beta_o\tau_o) - z_1 x_o \cos(z_o + \beta_o\tau_o)\rfloor \quad (42)$$

$$b_{qy} = b_{qa}\lfloor x_o \cos(z_o + \beta_o\tau_o) + x_1 \cos(z_o + \beta_o\tau_o) - z_1 x_o \sin(z_o + \beta_o\tau_o)$$

$$+ y_o \sin(z_o + \beta_o\tau_o) + y_1 \sin(z_o + \beta_o\tau_o) + z_1 y_o \cos(z_o + \beta_o\tau_o)\rfloor . \quad (43)$$

However, we are interested in deriving equations governing the particle motion on the slow time scale τ_1. Therefore we perform an average over the fast time scale. We denote this averaging process by < > brackets, for example, Eq. (20a) becomes

$$\langle(\frac{\partial}{\partial\tau}\gamma\frac{\partial x}{\partial\tau})_2\rangle = -\langle\lfloor e_{rx}(1-\beta_z)\rfloor_2\rangle + \langle(\beta_z b_{qy})_2\rangle . \quad (44)$$

For the left-hand side of Eq. (44) after averaging we obtain $(\partial^2 x_o/\partial\tau_1{}^2)$. Thus the second order equation becomes

$$\gamma_o \frac{\partial^2 x_o}{\partial\tau_1{}^2} = \langle\beta_{z,1}(e_{rx,1} + b_{qy,1})\rangle - \langle e_{rx,2}\rangle + \beta_o \langle e_{rx,2} + b_{qy,2}\rangle . \quad (45)$$

Inserting Eq. (43) into Eq. (45), using the fact, that $\beta_{z,1}$ and z_1 do not depend on τ_o and the assumption, that $\partial e_{rx}/\partial x_o = \partial e_{ry}/\partial y_o = 0$, we obtain

$$\gamma_o \frac{\partial^2 x_o}{\partial \tau_1^2} = \beta_o b_{qa} \langle x_1 \cos (z_o + \beta_o \tau_o) + y_1 \sin (z_o + \beta_o \tau_o) \rangle. \quad (46)$$

Analogous to Eq. (46) we can write an equation for the y-component

$$\gamma_o \frac{\partial^2 y_o}{\partial \tau_1^2} = \beta_o b_{qa} \langle x_1 \sin (z_o + \beta_o \tau_o) - y_1 \cos (z_o + \beta_o \tau_o) \rangle. \quad (47)$$

Making use of the expressions for x_1 and y_1 (Eqs. (40)-(41)) and averaging over the fast time scale we obtain

$$\frac{\partial^2 x_o}{\partial \tau_1^2} = \frac{e_{ra} b_{qa}}{\gamma_o^2 \kappa} \cos \left[(\kappa + 1) z_o + \phi \right] - \frac{b_{qa}^2}{\gamma_o^2} x_o$$

$$\frac{\partial^2 y_o}{\partial \tau_1^2} = \frac{e_{ra} b_{qa}}{\gamma_o^2 \kappa} \sin \left[(\kappa + 1) z_o + \phi \right] - \frac{b_{qa}^2}{\gamma_o^2} y_o. \quad (48)$$

In writing Eqs. (48), we have utilized the resonance condition, Eq. (28), which here is expressed as $\beta_o = \kappa/(\kappa + 1)$. Consider now the second order equation for Eq. (21c). The left-hand side of Eq. (21c) after averaging becomes $\gamma_o^3(\partial \beta_{z1}/\partial \tau_1)$, here we also used relation (34). Thus we can write, that

$$\gamma_o^3 \frac{\partial^2 z_o}{\partial \tau_1^2} = \langle \beta_{y,1}(b_{qy,1} - e_{ry,1}) - \beta_{x,1}(b_{qy,1} + e_{rx,1}) \rangle. \quad (49)$$

To obtain the expressions for $\beta_{y,1}$ and $\beta_{x,1}$ we differentiate Eqs. (40)-(41), respectively. Then utilizing Eqs. (36)-(37) and averaging over the fast time

scale we obtain the z-component of the equation of motion on the slow time scale, namely

$$\gamma_o^3 \frac{\partial^2 z_o}{\partial \tau_1{}^2} = \frac{e_{ra} b_{qa}}{\gamma_o \beta_o} \{ y_o \cos [(\kappa + 1) z_o + \phi]$$

$$- x_o \sin [(\kappa + 1) z_o + \phi] \}. \tag{50}$$

(Recall that z_o is not the unperturbed orbit, but is the orbit minus the resonant velocity times time, see Eq. (27).)

Denoting

$$a \equiv \frac{e_{ra} b_{qa}}{\gamma_o^2 \kappa},$$

$$a_z \equiv \frac{e_{ra} b_{qa}}{\gamma_o^4 \beta_o},$$

$$\alpha \equiv \frac{b_{qa}}{\gamma_o}, \tag{51}$$

the final set of equations of motion on the slow time scale is

$$\frac{d^2 x}{d\tau^2} = a \cos [(\kappa + 1) z + \phi] - \alpha^2 x \tag{52a}$$

$$\frac{d^2 y}{d\tau^2} = a \sin [(\kappa + 1) z + \phi] - \alpha^2 y \tag{52b}$$

$$\frac{d^2 z}{d\tau^2} = a_z \{ y \cos [(\kappa + 1) z + \phi] - x \sin [(\kappa + 1) z + \phi] \}. \tag{52c}$$

For convenience, we have dropped the subscripts o and 1, since for the following analyses we will use only the Eqs. (52). When necessary we will return to the original notation.

V. Linear Analyses

In order to solve the non-linear coupled system of Eqs. (52) we will employ a perturbation analysis. We assume that E_o (a, a_z) is small and the electron displacement and energy change can be expanded in powers of E_o.

For convenience of the analysis we introduce a complex variable $\xi = x + iy$, in which case Eqs. (52) become,

$$\frac{d^2\xi}{d\tau^2} = a \exp(i\chi) - \alpha^2\xi \tag{53}$$

$$\frac{d^2z}{d\tau^2} = \frac{a_z}{2i} \left[\xi \exp(-i\chi) - \xi^* \exp(i\chi)\right] \tag{54}$$

where $\chi = (\kappa + 1)z + \phi \equiv \tilde{\kappa}z + \phi$ and ξ^* is the complex conjugate of ξ. The equations to zero order in the radiation field are

$$\frac{d^2\xi^{(o)}}{d\tau^2} = -\alpha^2\xi^{(o)},$$

$$\frac{d^2z^{(o)}}{d\tau^2} = 0. \tag{55}$$

The general solution of the first, harmonic oscillator, equation is given by

$$\xi^{(o)} = \rho_+ e^{i(\alpha\tau + \theta_+)} + \rho_- e^{-i(\alpha\tau + \theta_-)} \tag{56}$$

where the real quantities ρ_+, ρ_-, θ_+ and θ_- are obtained from the initial

conditions. In Appendix A we derive these quantities in terms of the parameters describing the injection of the electron beam into the wiggler field. The oscillation corresponds to betatron oscillations in the strong focusing wiggler field.

The solution of the second equation in (55) is

$$z^{(o)} = z_i + (\beta_i - \beta_o)\tau \tag{57}$$

where z_i and β_i are the position of the electron along the z axis and its velocity at the moment ($\tau = 0$) of entering the interaction region, respectively. (Recall that $z^{(o)}$ in Eq. (57) varies slowly in τ. The actual fast variation is obtained by adding a $(\beta_o\tau)$ to $z^{(o)}$.)

We denote

$$\Delta\omega = \tilde{\kappa}(\beta_i - \beta_o)$$

using the resonance condition we can reexpress

$$\Delta\omega = \omega_r - (k_r + k_q)v_i.$$

Thus the zero order expression for χ becomes

$$\chi^{(o)} = \Phi_o + \Delta\omega\tau \tag{58}$$

where $\Phi_o = \tilde{\kappa}z_i + \phi$.

We now calculate ξ to first order in E_o. It obeys a forced oscillator equation with $\xi^{(1)} = d\xi^{(1)}/d\tau = 0$ at $\tau = 0$ as initial conditions, so it has a

form

$$\xi^{(1)} = a \exp(i\Phi_o)/(\alpha_+\alpha_-)\{\exp(i\Delta\omega\tau) - \frac{1}{2}\frac{\alpha_+}{\alpha}\exp(i\alpha\tau)$$

$$- \frac{1}{2}\frac{\alpha_-}{\alpha}\exp(-i\alpha\tau)\} \tag{59}$$

where $\alpha_\pm = \alpha \pm \Delta\omega$.

The first order expression for z is obtained by integrating Eq. (54) with $\xi = \xi^{(o)}$ and $\chi = \chi^{(o)}$, the result is given by

$$z^{(1)} = a_z\{\frac{\rho_-}{\alpha_+^2}[\sin(\alpha_+\tau + \Phi_o + \theta_-) - \sin(\theta_- + \Phi_o) - \alpha_+\tau\cos(\theta_- + \Phi_o)]$$

$$- \frac{\rho_+}{\alpha_-^2}[\sin(\alpha_-\tau - \Phi_o + \theta_+) - \sin(\theta_+ - \Phi_o) - \alpha_-\tau\cos(\theta_+ - \Phi_o)]\}. \tag{60}$$

The equation for the second order term in the expansion of z

$$\frac{d^2z^{(2)}}{d\tau^2} = -\frac{a_z}{2}\{\tilde{\kappa}z^{(1)}[\xi^{(o)}\exp(-i\chi^{(o)}) + \xi^{(o)*}\exp(i\chi^{(o)})]$$

$$+ [\xi^{(1)}\exp(-i\chi^{(o)}) - \xi^{(1)*}\exp(i\chi^{(o)})]\}. \tag{61}$$

The first two terms on the right-hand side are similar to those of a conventional free electron laser, in that only zero order transverse motion in the wiggler comes into play. However, the frequency of this term is shifted due to the betatron oscillations of the electrons. The second two terms on the right-hand side have no analog in the standard one-dimensional treatments. They describe a resonant beating of the first order transverse oscillation with longitudinal motion. These last two terms on the right-hand side of Eq.

(61) after inserting Eqs. (58)-(59) will take the following form

$$\frac{a_z a}{2\alpha\alpha_+\alpha_-}(\alpha_- \sin \alpha_+\tau - \alpha_+ \sin \alpha_-\tau).$$

Inserting Eqs. (58) and (60) into the first two terms in the right-hand side of Eqs. (61) will result in an expression which will include terms dependent on Φ_o. We assume that particles are injected uniformly in z, so that an average over Φ_o can be done. Thus, the average expression for the right-hand side is given by

$$\frac{a_z^2 \tilde{\kappa}\, \rho_+\rho_-}{2}\left(\frac{1}{\alpha_-^2} - \frac{1}{\alpha_+^2}\right) \sin (2\alpha\tau + \theta_+ + \theta_-)$$

$$- \left[\rho_+ \cos (\alpha_-\tau + \theta_+) + \rho_- \cos (\alpha_+\tau + \theta_-)\right]$$

$$\times \left[\frac{\rho_+}{\alpha_-^2} (\sin \theta_+ + \alpha_-\tau \cos \theta_+) - \frac{\rho_-}{\alpha_+^2} (\sin \theta_- + \alpha_+\tau \cos \theta_-)\right]$$

$$+ \left[\rho_+ \sin (\alpha_-\tau + \theta_+) - \rho_- \sin (\alpha_+\tau + \theta_-)\right]$$

$$\times \left[\frac{\rho_+}{\alpha_-^2} (\cos \theta_+ - \alpha_-\tau \sin \theta_+) + \frac{\rho_-}{\alpha_+^2} (\cos \theta_- - \alpha_+\tau \sin \theta_-)\right].$$

At that point we introduce an additional assumption by taking $\rho_- = 0$, this simplifies the last expression considerably. In Appendix A we show that this assumption represents initial conditions for an electron whose cylindrical radius remains constant in time and rotates in azimuthal angle with angular velocity α. An analysis utilizing more general initial conditions will be performed in the future.

Thus Eq. (61) reduces to the following

$$\frac{d^2 z^{(2)}}{d\tau^2} = \frac{a_z^2 \tilde{\kappa} \rho_+^2}{2\alpha_-^2} (\sin \alpha_-\tau - \alpha_-\tau \cos \alpha_-\tau)$$

$$+ \frac{a_z a}{2\alpha} \left(\frac{\sin \alpha_+\tau}{\alpha_+} - \frac{\sin \alpha_-\tau}{\alpha_-}\right). \tag{62}$$

Integrating Eq. (62) once we obtain

$$\beta_z^{(2)} = \frac{a_z a}{2\alpha} \left(\frac{1 - \cos \alpha_+\tau}{\alpha_+^2} - \frac{1 - \cos \alpha_-\tau}{\alpha_+^2}\right)$$

$$+ \frac{a_z^2 \tilde{\kappa} \rho_+^2}{\alpha_-^3} (1 - \cos \alpha_-\tau - \frac{1}{2} \alpha_-\tau \sin \alpha_-\tau). \tag{63}$$

VI. Single Pass Gain Calculation

Following the usual procedure[6] the gain G(t) is defined by

$$G(t) = \frac{\gamma_i - \bar{\gamma}(\tau)}{\gamma_i} \gamma_i mc^2 \frac{n_e V}{\frac{E_o^2}{4\pi} V} \tag{64}$$

where V is the volume of a section of the beam with number density n_e, $\bar{\gamma}(\tau)mc^2$ is the average over Φ_o of the electron energy.

The averaging of $\gamma = \gamma_o + \gamma_1^{(1)} + \gamma_1^{(2)} + \cdots$ over Φ_o eliminates the first order contribution in (64), therefore inserting Eq. (63) into $\gamma_1^{(2)} = \gamma_o^3 \beta_o \beta_{z,1}^{(2)}$ and the result into Eq. (64) yields the following expression for the gain

$$G(\tau) = \frac{4\pi n_e}{E_o^2} \gamma_o^3 mc^2 \beta_o \left\{\frac{a_z a}{2\alpha} \left[\frac{(1-\cos\alpha_-\tau)}{\alpha_-^2} \frac{(1-\cos\alpha_+\tau)}{\alpha_+^2}\right]\right.$$

$$\left. - \frac{a_z^2 \tilde{\kappa} \rho_+^2}{\alpha_-^3} \{1 - \cos\alpha_-\tau - \frac{1}{2} (\alpha_-\tau)\sin \alpha_-\tau\}\right\}. \tag{65}$$

Utilizing the definitions for a, a_z, α and δ we obtain

$$\hat{G}(\tau) = \left\{\frac{\beta_o^2}{8\alpha\tau}\left[\frac{(1-\cos\alpha_-\tau)}{(\alpha_-\tau)^2} - \frac{(1-\cos\alpha_+\tau)}{(\alpha_+\tau)^2}\right]\right\}$$

$$- \left\{\rho_+^2\gamma_o^2 \left(1 - \cos\alpha_-\tau - \frac{1}{2}\alpha_-\tau \sin\alpha_-\tau\right)/(\alpha_-\tau)^3\right\} \qquad (66)$$

where

$$\hat{G}(\tau) = G(\tau)/[4(\frac{\delta}{\alpha^2})\ \alpha^2(\alpha\tau)^3].$$

We denote the first curly bracket in Eq. (66) $\hat{G}_b(\tau)$ and the second $\hat{G}_\rho(\tau)$, thus $\hat{G} = \hat{G}_b + \hat{G}_\rho$. The expression for $\hat{G}_\rho(\tau)$ is similar to the formula for gain obtained in single particle analyses with a dipole wiggler (see Eq. (16) in Ref. (6)). The main difference is that the resonance is at $\Delta\omega = \alpha$ instead of $\Delta\omega = 0$. Thus the beat frequency resonates with the betatron frequency. Had we chosen instead the other initialization, $\rho_- \neq 0$, $\rho_+ = 0$, $\rho_- \neq 0$, the particles would rotate in the opposite sense and the resonant condition would have been $\Delta\omega = -\alpha$. $\hat{G}_b$ is an additional term associated with the transverse bunching in the FEL with a quadrupole wiggler, however it is smaller in magnitude than $\hat{G}_\rho(\tau)$. In Fig. 3 we show the $(\hat{G}_b/\beta_o^2)$ as a function of $(\underline{\alpha}\ \tau)$ for different $\alpha\tau$ and in Fig. 4 we present $(\hat{G}_p/\rho_+^2\gamma_o^2)$ as a function of $(\alpha_-\tau)$. A plot of the normalized gain is shown in Fig. 5. The maximum of $\hat{G}_\rho(\tau)$ appears at $\alpha_-\tau \approx 2.6$ and is given by,

$$G_{\rho,max} \approx 0.27\ (\frac{\delta}{\alpha^2})\alpha(\alpha\tau)^3(\rho_+\gamma_o^2).$$

This is nearly equal to the maximum value of $\hat{G}$. Defining $t = L/c\beta_o$, where L is the interaction region length, the maximum gain is given by

$$G_{max} = 0.27\ (e^4B_{oq}^2n_e\lambda_q)(\frac{N\ \lambda_q}{\gamma_o mc^2})^3\ (\frac{2\pi R_b}{\gamma_q})^2 \qquad (67)$$

where $N = L/\lambda_q$ is the number of wiggler periods in the interaction region.

From Eq. (67) we see that G_{max} for an FEL with a quadrupole pump has the same functional dependence as an FEL with a dipole pump, provided $B_{od} = B_{oq}(k_q R_b)$.

Conclusions

We have shown that relativistic electron trajectories in a rotating quadrupole periodic magnetic field are stable even if large space charge forces are present. This is not the case for the electron trajectories in a dipole periodic magnetic field. In the last case to improve the stability a constant magnetic field is introduced in addition to the periodic field. However, analyses have shown that the orbits are not stable for all initial conditions.[9,10] In the case of a quadrupole field the stability conditions and the theoretical maximum value for the electron beam current depends on the beam energy, density and radius and, in principle, can be close to the Alfvén-Lawson limiting current.

The linear, low gain analysis indicated that by utilizing a quadrupole periodic magnetic field as a wiggler in an FEL we obtain a positive gain. The maximum value of this gain has the same parametric dependence as for an FEL operating with the usual dipole wiggler, provided the same value of the amplitude of magnetic field at the beam position can be produced.

Therefore, an FEL with rotating quadrupole pump represents an interesting new concept to obtain high-power, coherent radiation in the millimeter and sub-millimeter regime.

In a future work we will investigate the feasibility of such devices by performing single-particle nonlinear analyses in Compton regime and an analysis of the device in Raman regime.

Acknowledgment

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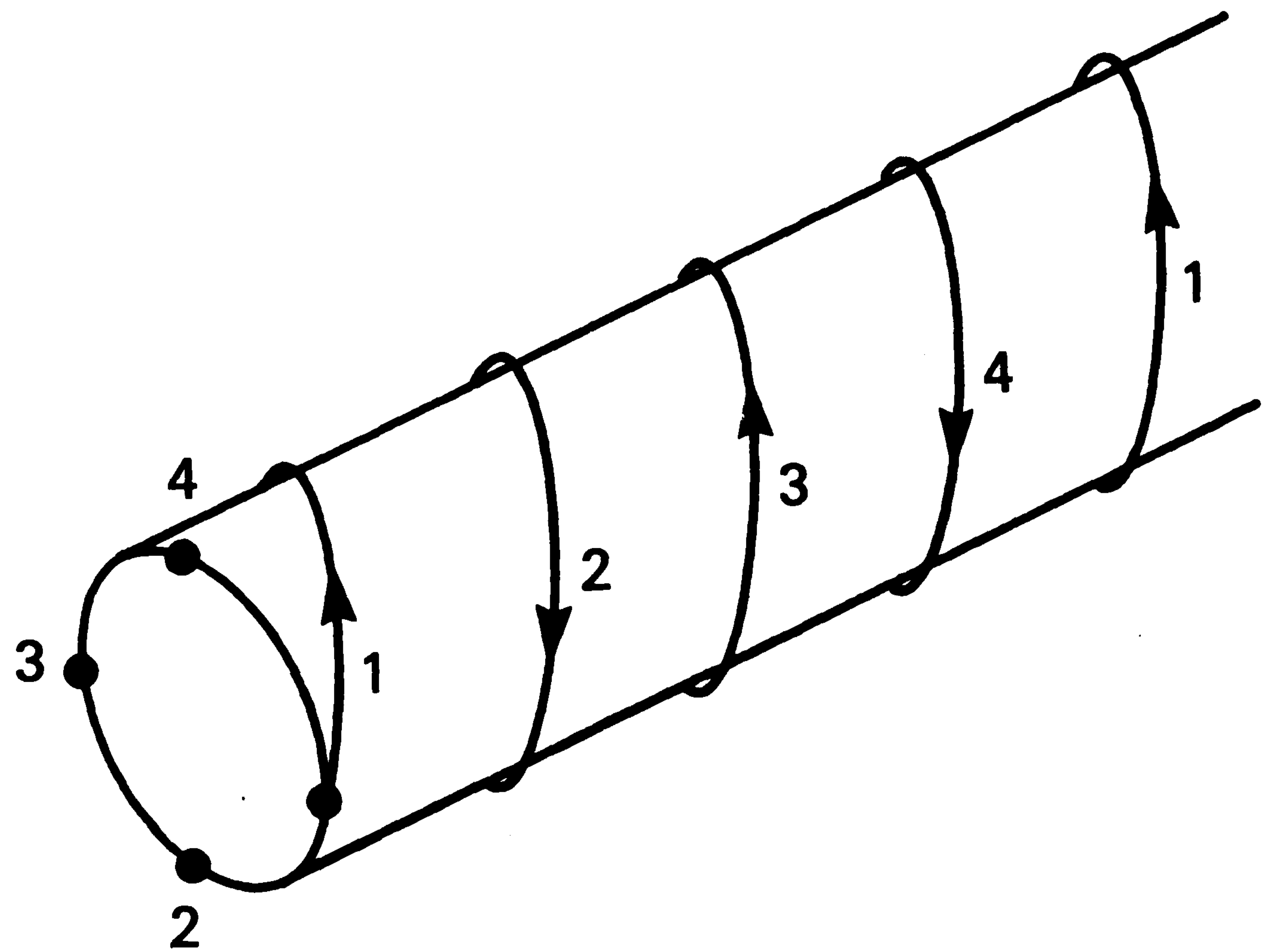

Fig. 1 Helical winding of four wires with current flowing in opposite directions to produce a continuously rotating quadrupole magnetic field.

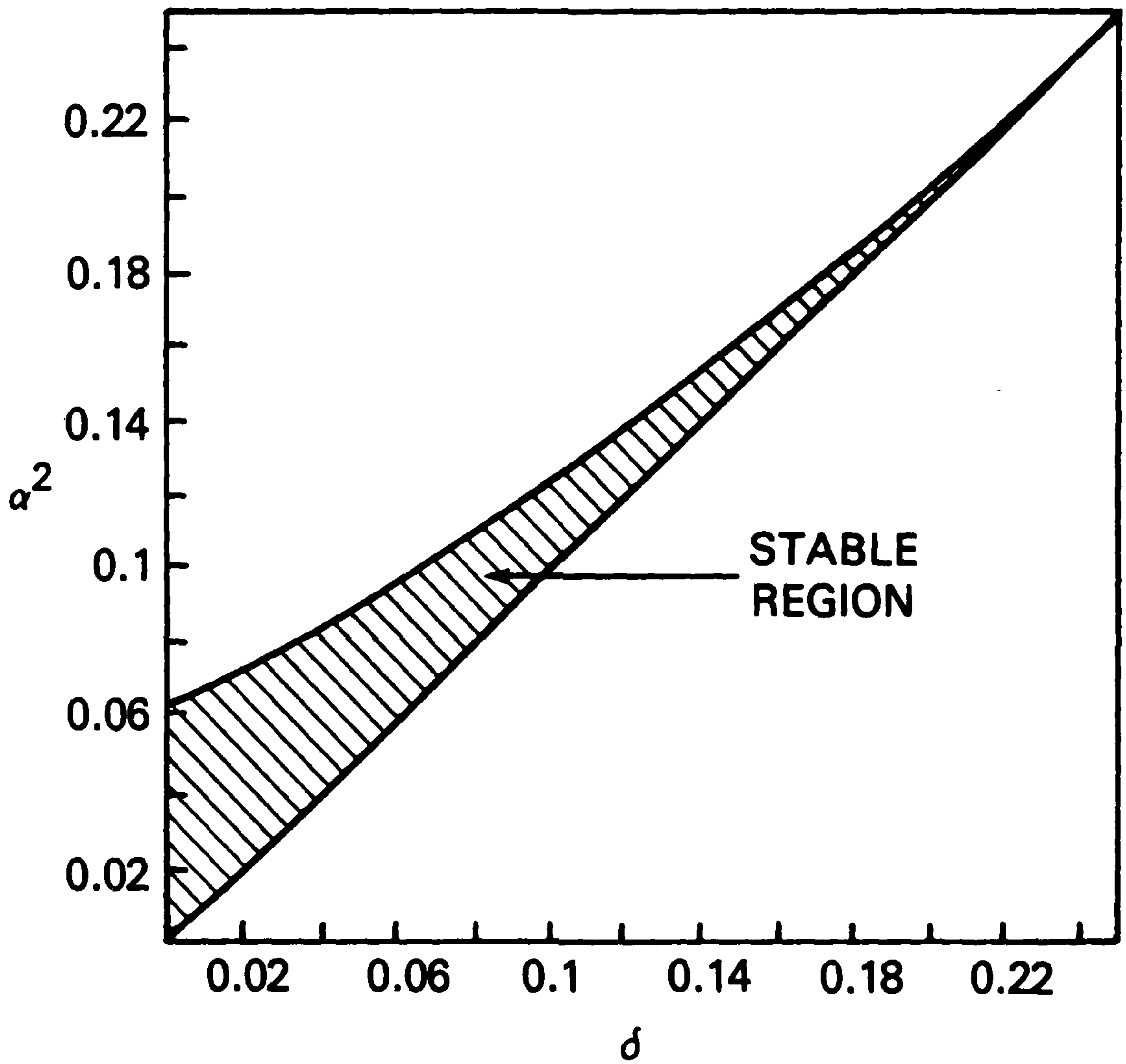

Fig. 2 Shows the stability region, where the upper line is $\alpha^2 = (\delta + 1/4)^2$ and the lower line is $\alpha^2 = \delta$, where $\alpha = (\Omega_q/\gamma_o k_q C)$ is normalized strength of the wiggler field and $\delta = \omega_b^2/(2\gamma_o^3\ \beta_o^2\ k_q^2 C^2)$ is normalized beam plasma oscillations.

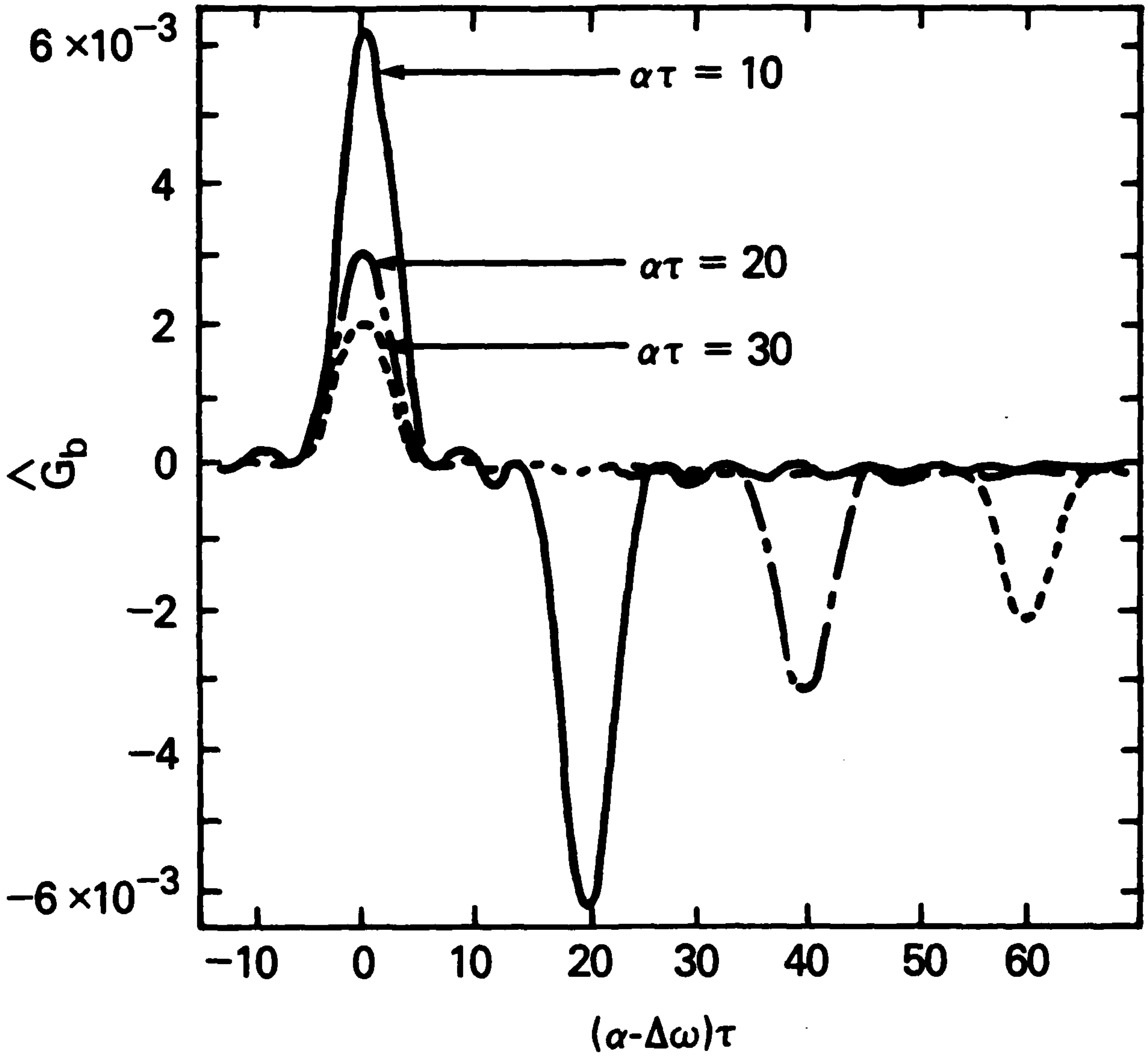

Fig. 3 Presents the dependence of $(\hat{G}_b/\beta_o^2)$ on $\alpha_{\_}\tau = (\alpha - \Delta\omega)\tau$, where $\hat{G}_b$ is the normalized gain produced by transverse bunching and $\Delta\omega = \kappa - \tilde{\kappa}\beta_{\perp}$. The solid, dashed-dotted and dotted lines shows $\hat{G}_b$ as a function of $\alpha\tau$ for $\alpha\tau = 10$, 20 and 30, respecively.

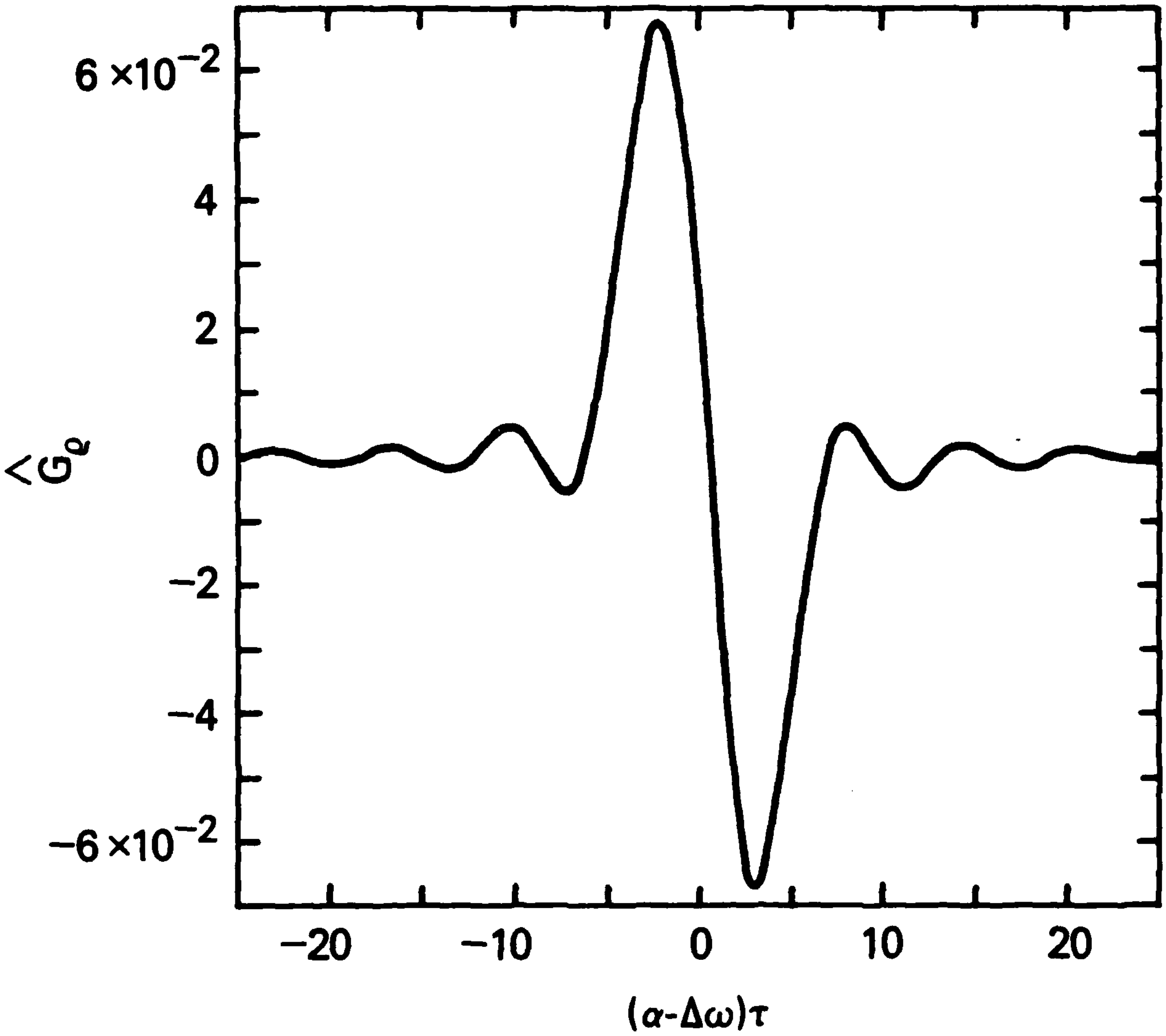

Fig. 4 Presents the dependence of $(\hat{G}_\rho/\rho_+^2\,\gamma_o^2)$ on $\alpha_-\tau$, where $\hat{G}_\rho$ is the normalized gain produced by longitudinal bunching and ρ_+ is the normalized beam radius.

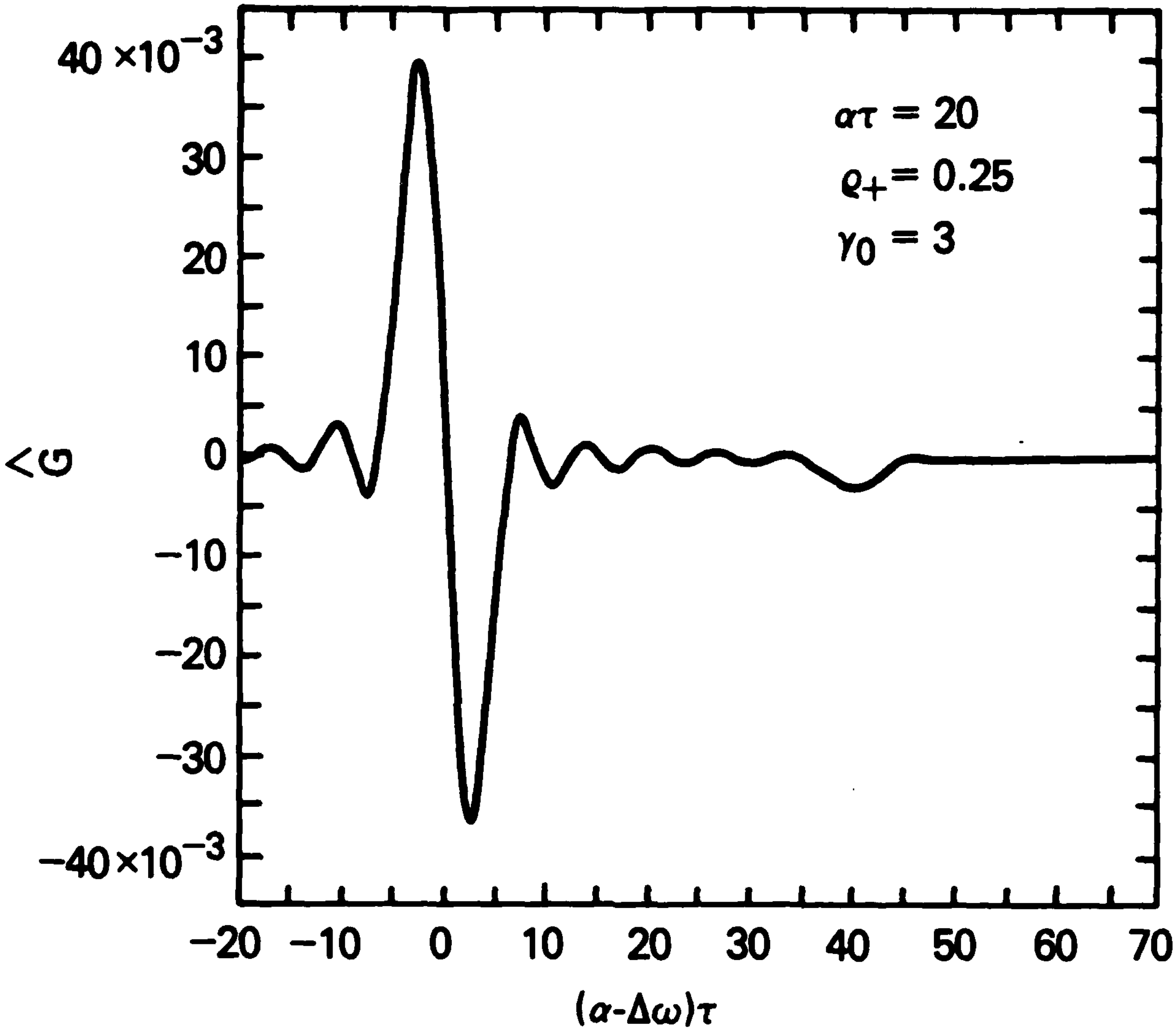

Fig. 5 Graph of normalized gain $\hat{G}$ as a function of $\alpha_{-}\tau$ for $\alpha\tau = 20$, $\rho_+ = 1/4$ and γ_o^3.

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Appendix A

We are interested in relating the parameters ρ_+, ρ_-, θ_+ and θ_- in Eq. (56) with the initial conditions of the electron beam injected into quadrupole periodic magnetic field. Electron position is described in cylindrical coordinates by its radius R and angle θ, radial velocity V_r and angular velocity V_θ. From Eq. (56) follows that

$$
\begin{aligned}
x^{(o)}(0) &= \rho_+\cos\theta_+ + \rho_-\cos\theta_- = R\cos\theta \\
y^{(o)}(0) &= \rho_+\sin\theta_+ - \rho_-\sin\theta_- = R\sin\theta \\
V_x^{(o)}(0) &= -\alpha(\rho_+\sin\theta_+ + \rho_-\sin\theta_-) = V_R\cos\theta - V_\theta\sin\theta \\
V_y^{(o)}(0) &= \alpha(\rho_+\cos\theta_+ - \rho_-\cos\theta_-) = V_R\sin\theta + V_\theta\cos\theta
\end{aligned}
\tag{A-1}
$$

then

$$
\begin{aligned}
\rho_+\cos\theta_+ &= \frac{1}{2}\left[\left(R + \frac{V_\theta}{\alpha}\right)\cos\theta + \frac{V_R}{\alpha}\sin\theta\right] \\
\rho_-\cos\theta_- &= \frac{1}{2}\left[\left(R - \frac{V_\theta}{\alpha}\right)\cos\theta - \frac{V_R}{\alpha}\sin\theta\right] \\
\rho_+\sin\theta_+ &= \frac{1}{2}\left[\left(R + \frac{V_\theta}{\alpha}\right)\sin\theta - \frac{V_R}{\alpha}\cos\theta\right] \\
\rho_-\sin\theta &= -\frac{1}{2}\left[\left(R - \frac{V_\theta}{\alpha}\right)\sin\theta + \frac{V_R}{\alpha}\cos\theta\right].
\end{aligned}
\tag{A-2}
$$

Introduce a following notation

$$R + \frac{V_\theta}{\alpha} = U_1 \cos\psi_1$$

$$\frac{V_R}{\alpha} = - U_1 \sin\psi_1 \qquad \text{(A-3)}$$

and

$$R - \frac{V_\theta}{\alpha} = U_2 \cos\psi_2$$

$$\frac{V_R}{\alpha} = U_2 \sin\psi_2 \qquad \text{(A-4)}$$

from (A-2) we obtain

$$\rho_+ \cos\theta_+ = \frac{1}{2} U_1 \cos(\theta + \psi_1)$$

$$\rho_+ \sin\theta_+ = \frac{1}{2} U_1 \sin(\theta + \psi_1)$$

$$\text{(A-5)}$$

$$\rho_- \cos\theta_- = \frac{1}{2} U_2 \cos(\theta + \psi_2)$$

$$\rho_- \sin\theta_- = - \frac{1}{2} U_2 \sin(\theta + \psi_2)$$

with constraint

$$U_2 \sin\psi_2 = - U_1 \sin\psi_1. \qquad \text{(A-6)}$$

Thus we can express ρ_+, ρ_-, θ_+ and θ_- in terms of R, θ, V_R and ψ_1, namely from (A-5) follows

$$\rho_+ = \frac{1}{2} U_1$$

$$\rho_- = \frac{1}{2} U_2$$

$$\theta_+ = \theta + \psi_1 \qquad \text{(A-7)}$$

$$\theta_- = -\theta - \psi_2.$$

For example, if $V_R = 0$ then $U_2 = 0$, therefore $\psi_1 = 0, \pm\pi, \cdots$. In this case $\rho_- = 0$, $V_\theta = \alpha R$, $\rho_+ = R$ and $\theta_+ = \theta$, which are the conditions for an electron orbit with constant radius.